Symmetry Type Graphs of Polytopes and Maniplexes

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Abstract. We extend the notion of symmetry type graphs of maps to include maniplexes and (abstract) polytopes, using them to study k-orbit maniplexes (where the automorphism group has k orbits on flags). In particular, we show that there are no fully-transitive k-orbit 3-maniplexes with $k > 1$ an odd number. We classify 3-orbit maniplexes and determine all face transitivities for 3- and 4-orbit maniplexes. Moreover, we give generators of the automorphism group of a maniplex, given its symmetry type graph. Finally, we extend these notions to oriented maniplexes, and we provide a classification of oriented 2-orbit maniplexes and a generating set for their orientation-preserving automorphism group.

1. Introduction

While abstract polytopes are a combinatorial generalisation of classical polyhedra and polytopes, maniplexes generalise maps on surfaces and (the flag graph of) abstract polytopes. The combinatorial structure of $(n-1)$-maniplexes and n-polytopes is completely determined by an edge-coloured n-valent graph with chromatic index n, often called the flag graph. In particular, maps correspond to 2-maniplexes [21]. The symmetry type graph of a map is the quotient of its flag graph under the action of the automorphism group. This notion is equivalent to the Delaney-Dress symbol, described in [6, 7, 13] and also used in [5]. In this paper we extend the notion of symmetry type graphs of maps to maniplexes (and polytopes). Given a maniplex, its symmetry type graph encapsulates all the information of the local configuration of the flag orbits under the action of the automorphism group of the maniplex.

Traditionally, the main focus of the study of maps and polytopes has been their symmetries. Of particular interest are the maps and polytopes that are regular (having the maximum degree of symmetry) and chiral (having the maximum degree of symmetry by rotation) [15, 16, 19]. Edge-transitive maps were studied in [20] by Širáň, Tucker and Watkins. Such maps have either 1, 2, or 4 orbits of flags under the action of the automorphism group. More recently Orbanić, Pellicer and Weiss extended this study and classified k-orbit maps (maps with k orbits of flags under the action of their automorphism group) up to $k \leq 4$ in [17]. Little is known about polytopes that are neither regular nor chiral. In [11] Hubard gives a complete characterisation of the automorphism groups of 2-orbit and fully-transitive polyhedra (that is, polyhedra whose automorphism group is transitive on the vertices, edges and faces). Moreover, she finds generators of the automorphism group of a 2-orbit polytope of any given rank.

Symmetry type graphs of the Platonic and Archimedean solids were determined in [14]. In [4] Del Río-Francos, Hubard, Orbanić and Pisanski determine the possible symmetry type graphs of 2-maniplexes of up to 5 vertices and give, for up to 7 vertices, the possible symmetry type graphs that a properly self-dual, an improperly self-dual, and a medial 2-maniplex might have. The possible symmetry type graphs that a truncation of a map can have is determined in [3]. A strategy for generating symmetry type graphs can be found in [1].

Using symmetry type graphs, we classify 3-orbit maniplexes and describe generators of their automorphism groups. In particular, we show that 3-orbit maniplexes are never fully-transitive, but they are i-face-transitive for all but one or two values of i, depending on the class. We extend further the study of symmetry type graphs to show that if a 4-orbit maniplex is not fully-transitive, then it is i-face-transitive for all but at most three values of i. Moreover, we show that a fully-transitive 3-maniplex (or 4-polytope) that is not regular cannot have an odd number of flag-orbits under the action of the automorphism group.

The main result of the paper is stated in Theorem 4, where we give generators for the automorphism group of a k-orbit maniplex with respect to a chosen base flag.

The paper is divided into six sections, organised in the following way. In Section 2, we review some basic theory of polytopes and maniplexes, and describe their respective flag graphs. In Section 3, we extend the concept of symmetry type graphs of maps to maniplexes, and describe some of their properties. In Section 4, we study symmetry type graphs of highly symmetric maniplexes. In particular, we classify the possible symmetry type graphs with 3 vertices, determine the possible face-transitivities that a 4-orbit maniplex can have, and study some properties of fully-transitive maniplexes of rank 3. In Section 5 we give generators of the automorphism group of a k-orbit maniplex. Finally, in Section 6, we define oriented and orientable maniplexes. We then define the oriented flag di-graph and symmetry type di-graph of an oriented maniplex, and use the latter to classify oriented 2-orbit maniplexes and give generators for their orientation-preserving automorphism group.

2. Abstract Polytopes and Maniplexes

2.1. Abstract Polytopes

In this subsection we briefly review the basic theory of abstract polytopes and their monodromy groups (for details we refer the reader to [16] and [12]).

An (*abstract*) *polytope of rank* n, or simply an n-*polytope*, is a partially ordered set $\mathcal{P}$ with a strictly monotone rank function with range $\{-1, 0, \dots, n\}$. An element of rank j is called a j-*face* of $\mathcal{P}$, and a face of rank 0, 1 or $n-1$ is called a *vertex*, *edge* or *facet*, respectively. A *chain of* $\mathcal{P}$ is a totally ordered subset of $\mathcal{P}$. The maximal chains, or *flags*, all contain exactly $n+2$ faces, including a unique least face F_{-1} (of rank -1) and a unique greatest face F_n (of rank n). The set of all flags of $\mathcal{P}$ shall be denoted by $\mathcal{F}(\mathcal{P})$. A polytope $\mathcal{P}$ has the following homogeneity property (*diamond condition*): whenever $F \leq G$, with F a $(j-1)$-face and G a $(j+1)$-face for some j, then there are exactly two j-faces H with $F \leq H \leq G$. Two flags are said to be *adjacent* (i-*adjacent*) if they differ in a single face (just in their i-face, respectively). The diamond condition can be rephrased by saying that every flag Φ of $\mathcal{P}$ has a unique i-adjacent flag, denoted Φ^i, for each $i = 0, \dots, n-1$. We extend this notation inductively and say that $\Phi^{i_0, i_1, \dots, i_k} = (\Phi^{i_0, i_1, \dots i_{k-1}})^{i_k}$, for $i_0, \dots i_k \in \{0, \dots, n-1\}$. Finally, $\mathcal{P}$ is *strongly flag-connected* in the sense that, if Φ and Ψ are two flags, then they can be joined by a sequence of successively adjacent flags, each containing $\Phi \cap \Psi$.

We note that if F is an i-face of $\mathcal{P}$, then we may naturally identify F with the poset $\{G \mid G \leq F\}$. This poset clearly satisfies all of the requirements to be a polytope, and so we may view the i-faces of $\mathcal{P}$ as i-polytopes themselves.

Let $\mathcal{P}$ be an abstract n-polytope and for each $i \in \{0, \dots, n-1\}$, let r_i be the element of $Sym(\mathcal{F}(\mathcal{P}))$ that sends each flag to its i-adjacent flag. That is, for each $\Phi \in \mathcal{F}(\mathcal{P})$,

$$\Phi^{r_i} = \Phi^i,$$

where Φ^{r_i} denotes the action of r_i on the flag Φ. The group $\langle r_0, r_1, \dots, r_{n-1} \rangle$ is called the *monodromy (or connection) group* of $\mathcal{P}$ (see for example [8, 12]), which we will denote by $\mathrm{Mon}(\mathcal{P})$. Because $\mathcal{P}$ satisfies the diamond condition, each of the permutations r_i is an involution. We furthermore observe that whenever $|i-j| \geq 2$, $\Phi^{i,j} = \Phi^{j,i}$, implying that in this case, the permutation $r_i r_j$ is also an involution. Hence, $\mathrm{Mon}(\mathcal{P})$ is a quotient of the universal string Coxeter group $[\infty, \dots, \infty] = \langle r_0, \dots, r_{n-1} \rangle$, whose only defining relations are $r_i^2 = \varepsilon$ and $(r_i r_j)^2 = \varepsilon$ whenever $|i-j| \geq 2$. Note that the connectivity of $\mathcal{P}$ immediately implies that $\mathrm{Mon}(\mathcal{P})$ acts transitively on $\mathcal{F}(\mathcal{P})$

An *automorphism* of a polytope $\mathcal{P}$ is a bijection of $\mathcal{P}$ that preserves the order. We shall denote the group of automorphisms of $\mathcal{P}$ by $\mathrm{Aut}(\mathcal{P})$. Note that any automorphism of $\mathcal{P}$ induces a bijection of its flags that preserves the i-adjacencies, for every $i \in \{0, 1, \dots, n-1\}$. A polytope $\mathcal{P}$ is said to be *regular* if the action of $\mathrm{Aut}(\mathcal{P})$ is regular on $\mathcal{F}(\mathcal{P})$. If $\mathrm{Aut}(\mathcal{P})$ has exactly 2 orbits on $\mathcal{F}(\mathcal{P})$ in such a way that adjacent flags belong to different orbits, $\mathcal{P}$ is called a *chiral polytope*. We say that a polytope is a *k-orbit polytope* if the action of $\mathrm{Aut}(\mathcal{P})$ has exactly k orbits on $\mathcal{F}(\mathcal{P})$. Hence, regular polytopes are 1-orbit polytopes and chiral polytopes are (one type of) 2-orbit polytopes.

Given an n-polytope $\mathcal{P}$, we define the *flag graph* $\mathcal{G}_{\mathcal{P}}$ of $\mathcal{P}$ as follows. The vertices of $\mathcal{G}_{\mathcal{P}}$ are the flags of $\mathcal{P}$, and we put an edge between two of them whenever the corresponding flags are adjacent. Hence $\mathcal{G}_{\mathcal{P}}$ is n-valent (i.e., every vertex of $\mathcal{G}_{\mathcal{P}}$ has exactly n incident edges; to reduce confusion we avoid the alternative terminology 'n-regular'). Furthermore, we can colour the edges of $\mathcal{G}_{\mathcal{P}}$ with n different colours as determined by the adjacencies of the flags of $\mathcal{P}$. That is, an edge of $\mathcal{G}_{\mathcal{P}}$ has colour i, if the corresponding flags of $\mathcal{P}$ are i-adjacent. In this way every vertex of $\mathcal{G}_{\mathcal{P}}$ has exactly one edge of each colour (see Figure 1).

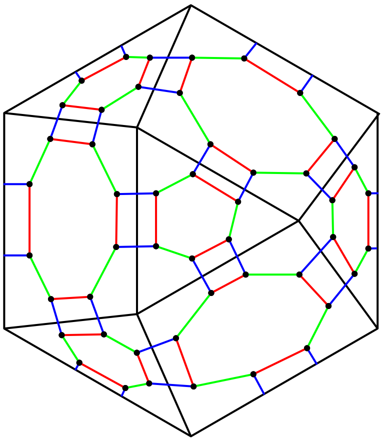


FIGURE 1. The flag graph of a cubeoctahedron.

It is straightforward to see that each automorphism of $\mathcal{P}$ induces an automorphism of the flag graph $\mathcal{G}_{\mathcal{P}}$ that preserves the colours. Conversely, every automorphism of $\mathcal{G}_{\mathcal{P}}$ that preserves the colours is a bijection of the flags that preserves all the adjacencies, inducing an automorphism of $\mathcal{P}$. In other words, the automorphism group $\mathrm{Aut}(\mathcal{P})$ of $\mathcal{P}$ is the colour preserving automorphism group $\mathrm{Aut}_p(\mathcal{G}_{\mathcal{P}})$ of $\mathcal{G}_{\mathcal{P}}$.

Note that the connectivity of $\mathcal{P}$ implies that the action of $\mathrm{Aut}(\mathcal{P})$ on $\mathcal{F}(\mathcal{P})$ is free (or semiregular). Hence, the action of $\mathrm{Aut}_p(\mathcal{G}_{\mathcal{P}})$ is free on the vertices of the graph $\mathcal{G}_{\mathcal{P}}$.

Since $r_i \in \mathrm{Mon}(\mathcal{P})$ takes each flag Φ to Φ^i, it is natural to label the edges of $\mathcal{G}_{\mathcal{P}}$ by the elements r_i instead of the colour i. Furthermore, to each walk along the edges of $\mathcal{G}_{\mathcal{P}}$, we can associate an element w of $\mathrm{Mon}(\mathcal{P})$. Hence, the connectivity of $\mathcal{P}$ implies that the action of $\mathrm{Mon}(\mathcal{P})$ is transitive on the vertices of $\mathcal{G}_{\mathcal{P}}$. Note that the i-faces of $\mathcal{P}$ can be regarded as the orbits of flags under the action of the subgroup $H_i = \langle r_j \mid j \neq i \rangle$. The i-faces of $\mathcal{P}$ can be also thought of as the connected components of the subgraph of $\mathcal{G}_\mathcal{P}$ obtained by deleting all the edges of colour i.

2.2. Maniplexes

Maniplexes were first introduced by Steve Wilson in [21], aiming to unify the notion of maps and polytopes. In this subsection we review their basic theory.

Let $\mathcal{M}$ consist of a set of *flags* $\mathcal{F}$ and a sequence of perfect matchings $(r_0, r_1, \dots, r_n)$, each of them partitioning the set $\mathcal{F}$ into parts of cardinality 2. Suppose that the partitions determined by r_i and r_j are disjoint for $i \neq j$; in other words, there are no flags such that r_i and r_j match them to the same flag. Then $\mathcal{M}$ is an *n-complex* if it is *connected* in the following way. Thinking of the n-complex $\mathcal{M}$ as the graph $\mathcal{G}_\mathcal{M}$ with vertex set $\mathcal{F}$, and with edges of colour i corresponding to the matching induced by r_i, we ask for the graph $\mathcal{G}_\mathcal{M}$ to be connected. Note that, in $\mathcal{G}_\mathcal{M}$, each r_i represents a matching of the graph. Hence, the subgraph of $\mathcal{G}_\mathcal{M}$ with edges of two given colours i and j form a 2-factor of the graph.

Each of the matchings r_i induces a permutation of $\mathcal{F}$ in a natural way, by sending each flag Φ to the flag that r_i matches to Φ, which we shall denote by Φ^i. (We will abuse notation and also use r_i to denote the permutation induced by r_i.) Note that we then have that $(\Phi^i)^i = \Phi$ for every flag Φ, implying that r_i, viewed as an element of $Sym(\mathcal{F})$, is an involution. Hence, the group $\mathrm{Mon}(\mathcal{M})$ generated by the permutations $r_0, r_1, \dots, r_n$ is the quotient of a Coxeter group. We say that $\mathcal{M}$ is an *n-maniplex* whenever the group $\mathrm{Mon}(\mathcal{M})$ is in fact the quotient of a string Coxeter group. In other words, a maniplex is a complex such that the permutations $r_i r_j$ are involutions whenever $|i-j| \geq 2$. Equivalently, if the graph $\mathcal{G}_\mathcal{M}$ is such that the 2-factors of colours i and j are squares for all $i, j \in \{0, \dots, n\}$ such that $|i-j| > 1$, then the complex $\mathcal{M}$ is in fact a maniplex. The group $\mathrm{Mon}(\mathcal{M})$ is called the *connection group* of the maniplex and $r_0, r_1, \dots, r_n$ are its *distinguished generators*. Note that, since $r_i r_j = r_j r_i$ whenever $|i-j| \geq 2$, then for any flag Φ of $\mathcal{M}$ and $i, j \in \{0, \dots, n\}$ such that $|i-j| \geq 2$ we have that $\Phi^{i,j} = \Phi^{r_i r_j} = \Phi^{r_j r_i} = \Phi^{j,i}$.

A 0-maniplex must be a graph with two vertices joined by an edge of colour 0. A 1-maniplex is associated to a 2-polytope or l-gon, whose graph contains $2l$ vertices joined by a perfect matching of colour 0 and a perfect matching of colour 1, each matching of size l. Every 2-maniplex induces a map (a 2-cell embedding of a connected graph on a surface). Furthermore, a map such that itself and its dual have no vertex of degree 1 induces a 2-maniplex. Any $(n+1)$-polytope can be thought as an n-maniplex. However, some maniplexes do not induce polytopes (see [21]).

An *automorphism* α of an n-maniplex is a colour-preserving automorphism of the graph $\mathcal{G}_\mathcal{M}$. Hence, α can be seen as a permutation of the flags in $\mathcal{F}$ that commutes with each of the permutations in the connection group. In analogy with polytopes, the connectivity of the graph $\mathcal{G}_\mathcal{M}$ implies that the action of the automorphism group $\mathrm{Aut}(\mathcal{M})$ of $\mathcal{M}$ is free on the vertices of $\mathcal{G}_{\mathcal{M}}$.

To have consistent concepts and notation between polytopes and maniplexes, we shall say that an i-face (or a face of rank i) of a maniplex is a connected component of the subgraph of $\mathcal{G}$ obtained by removing the i-edges of $\mathcal{G}$. Furthermore, we say that two flags Φ and Ψ are i-adjacent if $\Phi^{r_i} = \Psi$ (note that since r_i is an involution, $\Phi^{r_i} = \Psi$ implies that $\Psi^{r_i} = \Phi$, so the concept is symmetric).

Just as each i-face of a polytope corresponds to an i-polytope, we can associate an $(i-1)$-maniplex $\mathcal{M}_F$ to each i-face F of $\mathcal{M}$ by identifying two flags of F whenever there is a path between them consisting of edges with colours in $\{i+1, \dots, n\}$ (and then identifying any edges between each pair of points if those edges have the same colour). Equivalently, we can remove from F all edges of colours $\{i+1, \dots, n\}$, and then take one of the connected components. In fact, since $\langle r_0, \dots, r_{i-1} \rangle$ commutes with $\langle r_{i+1}, \dots, r_n \rangle$, the connected components of this subgraph of F are all isomorphic, so it does not matter which one we pick.

If Φ is a flag of $\mathcal{M}$ that contains the i-face F, then it naturally induces a flag $\overline{\Phi}$ in $\mathcal{M}_F$. Similarly, if $\varphi \in \mathrm{Aut}(\mathcal{M})$ fixes F, then φ induces an automorphism $\overline{\varphi} \in \mathrm{Aut}(\mathcal{M}_F)$, defined by $\overline{\Phi}\overline{\varphi} = \overline{\Phi\varphi}$. To check that this is well-defined, suppose that $\overline{\Phi} = \overline{\Psi}$; we want to show that $\overline{\Phi\varphi} = \overline{\Psi\varphi}$. Since $\overline{\Phi} = \overline{\Psi}$, it follows that $\Psi = \Phi^w$ for some $w \in \langle r_{i+1}, \dots, r_n \rangle$. Then $\Psi\varphi = (\Phi^w)\varphi = (\Phi\varphi)^w$, so that $\overline{\Psi\varphi} = \overline{\Phi\varphi}$.

By definition, the edges of $\mathcal{G}$ of one given colour form a perfect matching. The *2-factors* of the graph $\mathcal{G}$ are the subgraphs spanned by the edges of two different colours of edges.

Since the automorphisms of $\mathcal{M}$ preserve the adjacencies between the flags, it is not difficult to see that the following lemma holds.

Lemma 1. *Let Φ be a flag of $\mathcal{M}$ and let $a \in \mathrm{Mon}(\mathcal{M})$. If $\mathcal{O}_1$ and $\mathcal{O}_2$ denote the flag orbits of Φ and Φ^a (under $\mathrm{Aut}(\mathcal{M})$), respectively, then $\Psi \in \mathcal{O}_1$ if and only if $\Psi^a \in \mathcal{O}_2$.*

We say that a maniplex $\mathcal{M}$ is *i-face-transitive* if $\mathrm{Aut}(\mathcal{M})$ is transitive on the faces of rank i. We say that $\mathcal{M}$ is *fully-transitive* if it is i-face-transitive for every $i = 0, \dots, n$.

If $\mathrm{Aut}(\mathcal{M})$ has k orbits on the flags of $\mathcal{M}$, we say that $\mathcal{M}$ is a *k-orbit* maniplex. A 1-orbit maniplex is also called a *reflexible* maniplex. A 2-orbit maniplex with adjacent flags belonging to different orbits is a *chiral* maniplex. If a maniplex has at most 2 orbits of flags, and if its graph $\mathcal{G}$ is a bipartite graph such that each part is contained in an orbit, then the maniplex is said to be *rotary.*

As it should be clear, it makes no difference whether we consider an abstract n-polytope or an $(n-1)$-maniplex. Hence, though we will consider maniplexes throughout the paper, similar results will apply to polytopes.

3. Symmetry type graphs of polytopes and maniplexes

In this section we shall define the symmetry type graph of a maniplex. To this end, we shall make use of quotients of graphs. Therefore, we now consider pregraphs; that is, graphs that allow multiple edges and semi-edges.

Given an edge-coloured graph $\mathcal{G}$, and a partition $\mathcal{B}$ of its vertex set V, the *coloured quotient with respect to* $\mathcal{B}$, $\mathcal{G}_{\mathcal{B}}$, is defined as the pregraph with vertex set $\mathcal{B}$, such that for any two vertices $B, C \in \mathcal{B}$, there is an edge of colour a from B to C if and only if there exists $u \in B$ and $v \in C$ such that there is an edge of colour a from u to v. Edges between vertices in the same part of the partition $\mathcal{B}$ quotient into semi-edges (edges with exactly one end point).

Throughout the remainder of this section, let $\mathcal{M}$ be an $(n-1)$-maniplex and $\mathcal{G}_{\mathcal{M}}$ its coloured flag graph.

As we discussed in the previous section, $\mathrm{Aut}(\mathcal{M})$ acts semiregularly on the vertices of $\mathcal{G}_{\mathcal{M}}$. We shall consider the orbits of the vertices of $\mathcal{G}_{\mathcal{M}}$ under the action of $\mathrm{Aut}(\mathcal{M})$ as our partition $\mathcal{B}$, and denote $\mathcal{B} := \mathcal{O}rb$. Note that since the action is semiregular, every two orbits $B, C \in \mathcal{O}rb$ have the same number of elements. The *symmetry type graph* $T(\mathcal{M})$ of $\mathcal{M}$ is the coloured quotient graph of $\mathcal{G}_{\mathcal{M}}$ with respect to $\mathcal{O}rb$.

Since the flag graph $\mathcal{G}_{\mathcal{M}}$ is an undirected graph, then $T(\mathcal{M})$ is a pre-graph without loops or directed edges. Furthermore, as we are taking the coloured quotient, and $\mathcal{G}_{\mathcal{M}}$ is edge-coloured with n colours, Lemma 1 implies that $T(\mathcal{M})$ is an n-valent pre-graph, with one edge or semi-edge of each colour at each vertex. It is hence not difficult to see that if $\mathcal{M}$ is a reflexible maniplex, then $T(\mathcal{M})$ is a graph consisting of only one vertex and n semi-edges, all of them of different colours. In fact, the symmetry type graph of a k-orbit maniplex has precisely k vertices. Given vertices u, v of $T(\mathcal{M})$, if there is an i-edge joining them, we shall denote that edge as $(u, v)_i$, and we may denote v by u^i. Similarly, $(v, v)_i$ shall denote the semi-edge of colour i incident to the vertex v. Figure 2 shows the symmetry type graph of a reflexible 2-maniplex (on the left), and the symmetry type graph of the cuboctahedron: the quotient graph of the flag graph in Figure 1 with respect to the automorphism group of the cuboctahedron.

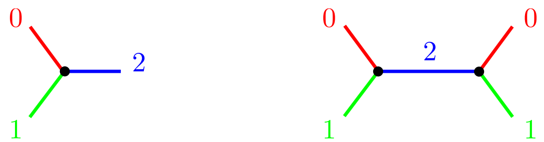


FIGURE 2. Symmetry type graphs of a reflexible 2-maniplex (on the left) and of the cuboctahedron (on the right).

Note that by the definition of $T(\mathcal{M})$, there exists a surjective function

$$\psi : V(\mathcal{G}_{\mathcal{M}}) \to V(T(\mathcal{M}))$$

that assigns, to each vertex of $V(\mathcal{G}_{\mathcal{M}})$, its corresponding orbit in $T(\mathcal{M})$. Hence, given $\Phi, \Psi \in V(\mathcal{G}_{\mathcal{M}})$, we have that $\psi(\Phi) = \psi(\Psi)$ if and only if Φ and Ψ are in the same orbit under $\mathrm{Aut}(\mathcal{M})$.

Fix a base flag Φ of $\mathcal{M}$ and let $N = \mathrm{Stab}_{\mathrm{Mon}(\mathcal{M})}(\Phi)$ and $\mathcal{N} = \mathrm{Norm}_{\mathrm{Mon}(\mathcal{M})}(N)$. Then there is a bijection between the cosets of $\mathcal{N}$ in $\mathrm{Mon}(\mathcal{M})$ and the set of orbits $\mathcal{O}rb$, given by sending each $\mathcal{N}w$ to the orbit of Φ^w (see [11]). Therefore, $T(\mathcal{M})$ is isomorphic to the graph of cosets of $\mathcal{N}$ in $\mathrm{Mon}(\mathcal{M})$ with respect to the generating set $r_0, r_1, \ldots, r_n$. That is, $T(\mathcal{M})$ is isomorphic to the graph whose vertices are the cosets of $\mathcal{N}$ in $\mathrm{Mon}(\mathcal{M})$, where there is an edge of colour i between two cosets $\mathcal{N}w$ and $\mathcal{N}v$ whenever $\mathcal{N}v = \mathcal{N}wr_i$.

Because of Lemma 1, we can define the action of $\mathrm{Mon}(\mathcal{M})$ on the vertices of $T(\mathcal{M})$. In fact, given $v \in T(\mathcal{M})$ and $a \in \mathrm{Mon}(\mathcal{M})$, then $v^a := \psi(\Phi^a)$, for any $\Phi \in \psi^{-1}(v)$. Since $\mathrm{Mon}(\mathcal{M})$ is transitive on the vertices of $\mathcal{G}_{\mathcal{M}}$, then it is also transitive on the vertices of $T(\mathcal{M})$, implying that $T(\mathcal{M})$ is a connected graph. Furthermore, the action of each generator r_i of $\mathrm{Mon}(\mathcal{M})$ on a vertex v of $T(\mathcal{M})$ corresponds precisely to the (semi-)edge of colour i incident to v. Hence, if we pick any flag $\Psi \in \psi^{-1}(v)$ and look at the set of flags that contain the same i-face F as Ψ does, the image of this set under ψ will be the orbit of v under $\langle r_j \mid j \neq i \rangle$. Therefore, the connected components of the subgraph $T^i(\mathcal{M})$ of $T(\mathcal{M})$ with edges of colours in $\{0, \ldots, n-1\} \setminus \{i\}$ correspond to the orbits of the i-faces under $\mathrm{Aut}(\mathcal{M})$. In particular this implies the following proposition.

Proposition 1. *Let $\mathcal{M}$ be a maniplex, with symmetry type graph $T(\mathcal{M})$. Let $T^i(\mathcal{M})$ be the subgraph of $T(\mathcal{M})$ obtained by erasing the i-edges of $T(\mathcal{M})$. Then $\mathcal{M}$ is i-face-transitive if and only if $T^i(\mathcal{M})$ is connected.*

In light of Proposition 1, we say that a symmetry type graph T is i-face-transitive if T^i is connected, and that T is a fully-transitive symmetry type graph if it is i-face-transitive for all i.

Recall that to each i-face F of $\mathcal{M}$, there is an associated $(i-1)$-maniplex $\mathcal{M}_F$. The symmetry type graph $T(\mathcal{M}_F)$ is related in a natural way to the connected component of $T^i(\mathcal{M})$ that corresponds to F:

Proposition 2. *Let F be an i-face of the maniplex $\mathcal{M}$, and let $\mathcal{M}_F$ be the corresponding $(i-1)$-maniplex. Let $\mathcal{C}$ be the connected component of $T^i(\mathcal{M})$ corresponding to F. Then there is a surjective function $\pi : V(\mathcal{C}) \to V(T(\mathcal{M}_F))$. Furthermore, if $j < i$ then each j-edge $(u, u^j)_j$ of $\mathcal{C}$ yields a j-edge or j-semi-edge $(\pi(u), \pi(u^j))_j$ in $T(\mathcal{M}_F)$ (and each j-semi-edge induces a j-semi-edge), and if $j > i$, then $\pi(u) = \pi(u^j)$.*

Proof. First, let Φ and Ψ be flags of $\mathcal{M}$ that are both in the connected component corresponding to F, and suppose that they lie in the same flag orbit, so that $\Psi = \Phi\varphi$ for some $\varphi \in \mathrm{Aut}(\mathcal{M})$. Then the induced automorphism $\overline{\varphi}$ of $\mathcal{M}_F$ sends $\overline{\Phi}$ to $\overline{\Psi}$, and therefore $\overline{\Phi}$ and $\overline{\Psi}$ lie in the same orbit. Furthermore, every flag of $\mathcal{M}_F$ is of the form $\overline{\Phi}$ for some Φ in F. Thus, each orbit of $\mathcal{M}$ that intersects F induces an orbit of $\mathcal{M}_F$, and it follows that there is a surjective function $\pi : V(\mathcal{C}) \to V(T(\mathcal{M}_F))$.

Consider an edge (or semi-edge) $(u, u^j)_j$ in $\mathcal{C}$. Then $u = \psi(\Phi)$ for some flag Φ in F, and we can take $u^j = \psi(\Phi^j)$. Both Φ and Φ^j induce flags in $\mathcal{M}_F$. If $j < i$, then $\overline{\Phi^j} = \overline{\Phi}^j$. Therefore, there must be a j-edge (or j-semi-edge) from the orbit of $\overline{\Phi}$ to the orbit of $\overline{\Phi^j}$; in other words, a j-edge (or j-semi-edge) from $\pi(u)$ to $\pi(u^j)$. On the other hand, if $j > i$, then $\overline{\Phi^j} = \overline{\Phi}$, and so $\overline{\Phi}$ and $\overline{\Phi^j}$ lie in the same orbit and thus $\pi(u) = \pi(u^j)$. □

Note that the (semi-)edges of a given colour i of $T(\mathcal{M})$ form a perfect matching (if we consider a semi-edge to match a vertex to itself). Given two colours i and j, the subgraph of $T(\mathcal{M})$ consisting of all the vertices of $T(\mathcal{M})$ and only the i- and j-(semi-)edges shall be called an (i, j) 2-factor of $T(\mathcal{M})$. Because $r_i r_j = r_j r_i$ whenever $|i - j| \geq 2$, in $\mathcal{G}_{\mathcal{M}}$, the alternating cycles of colours i and j have length 4. By Lemma 1 each of these 4-cycles should then factor, in $T(\mathcal{M})$, into one of the five graphs in Figure 3. Hence, if $|i-j| \geq 2$, then the connected components of the (i, j) 2-factors of $T(\mathcal{M})$ are precisely among these graphs.

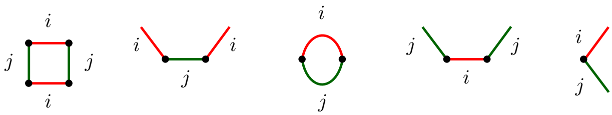


FIGURE 3. Possible quotients of $i - j$ coloured 4-cycles.

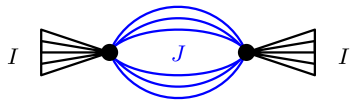
In light of the above observations we state the following lemma.

Lemma 2. *Let $T(\mathcal{M})$ be the symmetry type graph of a maniplex. If there are three distinct vertices $u, v, w \in V(T(\mathcal{M}))$ such that $(u, v)_i, (v, w)_j \in E(T(\mathcal{M}))$ with $|i-j| \geq 2$, then the connected component of the (i, j) 2-factor that contains v has four vertices.*

4. Symmetry type graphs of highly symmetric maniplexes

One can classify maniplexes with a small number of flag orbits (under the action of the automorphism group of the maniplex) in terms of their symmetry type graphs. Furthermore, given a symmetry type graph, one can read from the appropriate coloured subgraphs the different types of face transitivities that the maniplex has. To determine the possible symmetry types of a k-orbit $(n-1)$-maniplex, we need only consider connected, n-valent pre-graphs on k vertices such that they are properly n-edge-colourable, and such that whenever $|i - j| \geq 2$, the connected components of the 2-factors with colours i and j are one of the possibilities in Figure 3. Note, however, that it is unclear whether any pre-graph that satisfies these necessary conditions actually occurs as the symmetry type graph of a k-orbit $(n-1)$-maniplex.

As pointed out before, the symmetry type graph of a reflexible $(n-1)$-maniplex consists of one vertex and n semi-edges. The classification of two-orbit maniplexes in terms of the local configuration of their flags follows immediately from the possible symmetry type graphs. In fact, for each n, there are $2^n - 1$ possible symmetry type graphs with 2 vertices and n (semi)-edges, since given any proper subset I of the colours $\{0, 1, \dots, n-1\}$, there is a s graph with two vertices, $|I|$ semi-edges corresponding to the colours of I incident to each vertex, and where all the edges between the two vertices use the colours not in I (see Figure 4). This symmetry type graph corresponds precisely to polytopes in class 2_I (see [10]).



$I \subset \{0, 1, \dots, n-1\}$, $\qquad J = \{0, 1, \dots, n-1\} \setminus I$

FIGURE 4. The symmetry type graph of a maniplex in class 2_I.

Highly symmetric maniplexes can be regarded as those with few flag orbits or those with many face transitivities. In [4] one can find the complete list of possible symmetry type graphs of 2-maniplexes with at most 5 vertices. In this section we classify the possible symmetry type graphs with 3 vertices (in any rank) and study some properties of symmetry type graphs of 4-orbit maniplexes and fully-transitive 3-maniplexes.

4.1. Symmetry type graphs of 3-orbit maniplexes

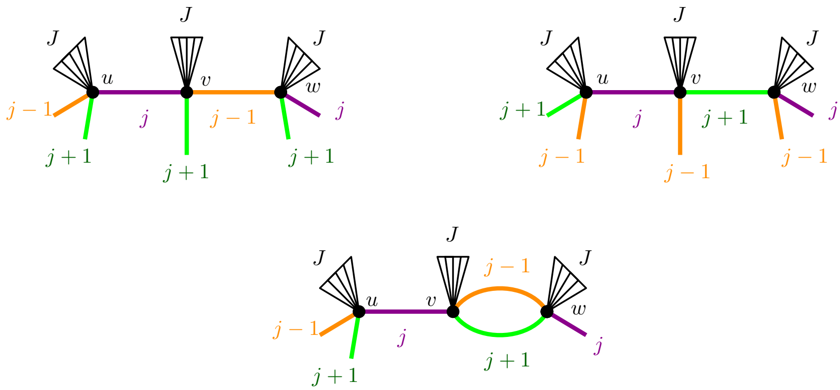
Here we will describe all possible symmetry type graphs of 3-orbit maniplexes and determine for which ranks j they are j-face-transitive.

Proposition 3. *The symmetry type graph of a 3-orbit maniplex of rank $n-1$ is of one of the forms in Figure 5. In particular, there are $2n-3$ different possible symmetry type graphs of 3-orbit maniplexes of rank $n-1$.*

Proof. Let $\mathcal{M}$ be a 3-orbit $(n-1)$-maniplex and $T(\mathcal{M})$ its symmetry type graph. Then, $T(\mathcal{M})$ is an n-valent properly edge-coloured graph with vertices v_1, v_2 and v_3. Recall that the set of colours $\{0, 1, \dots, n-1\}$ correspond to the distinguished generators $r_0, r_1, \dots, r_{n-1}$ of the connection group of $\mathcal{M}$, and that by $(u, v)_i$ we mean the edge between vertices u and v of colour i.

Since $T(\mathcal{M})$ is a connected graph, without loss of generality, we can suppose that there is at least one edge joining v_1 and v_2 and another joining v_2 and v_3. Let $j, k \in \{0, 1, \dots, n-1\}$ be the colours of these edges, respectively. That is, without loss of generality we may assume that $(v_1, v_2)_j$ and $(v_2, v_3)_k$ are edges of $T(\mathcal{M})$. By Lemma 2, we must have that $k = j \pm 1$, as otherwise $T(\mathcal{M})$ would have to have at least 4 vertices. This implies that, up to graph isomorphism, the only edges of $T(\mathcal{M})$ are either $(v_1, v_2)_j$ and $(v_2, v_3)_{j+1}$, $(v_1, v_2)_j$ and $(v_2, v_3)_{j-1}$ or $(v_1, v_2)_j$, $(v_2, v_3)_{j+1}$ and $(v_2, v_3)_{j-1}$, with $j \in \{1, 2, \dots, n-2\}$ (see Figure 5).

An easy computation now shows that there are $2n-3$ possible different symmetry type graphs of 3-orbit maniplexes of rank $n-1$. $\square$



$J = \{0, 1, \ldots, n-1\} \setminus \{j-1, j, j+1\}$

FIGURE 5. Possible symmetry type graphs of 3-orbit $(n-1)$-maniplexes with edges of colours $j-1$, j, and $j+1$, with $j \in \{1, 2, \ldots, n-2\}$.

Given a 3-orbit $(n-1)$-maniplex $\mathcal{M}$ with symmetry type graph having exactly two edges e and e' of colours j and $j+1$, respectively, for some $j \in \{0, \ldots, n-2\}$ (as in the upper two graphs of Figure 5), we shall say that $\mathcal{M}$ is in class $3^{j,j+1}$. If, on the other hand, the symmetry type graph of $\mathcal{M}$ has edges of colours $j-1$, j, and $j+1$, such that the edges of colours $j-1$ and $j+1$ are parallel (as in the lower graph of Figure 5), then we say that $\mathcal{M}$ is in class 3^j. From Figure 5 and Proposition 1, we observe that a maniplex in class $3^{j,j+1}$ is i-face-transitive whenever $i \neq j, j+1$, while a maniplex in class 3^j is i-face-transitive for every $i \neq j$. From here, we have the following results.

Proposition 4. *A* 3*-orbit maniplex is* j*-face-transitive if and only if it does not belong to any of the classes* 3^j, $3^{j,j+1}$ *or* $3^{j-1,j}$.

Theorem 1. *There are no fully-transitive 3-orbit maniplexes.*

Using Proposition 2, we get some information about the number of flag orbits that the j-faces have:

Proposition 5. *A* 3*-orbit maniplex in class* 3^j *or* $3^{j,j+1}$ *has reflexible* j*-faces.*

Proof. If $\mathcal{M}$ is a 3-orbit maniplex, then the orbits of the j-faces correspond to the connected components of $T^j(\mathcal{M})$. Assuming that $\mathcal{M}$ is in class 3^j or $3^{j,j+1}$, the graph $T^j(\mathcal{M})$ has two connected components; an isolated vertex, and two vertices that are connected by a $(j+1)$-edge (and a $(j-1)$-edge, if $\mathcal{M}$ is in class $3^{j,j+1}$). Then by Proposition 2, the j-faces that correspond to the isolated vertex are reflexible (that is, 1-orbit), and the edge with label $j+1$

forces an identification between the two vertices of the second component, so the j-faces in that component are also reflexible. $\square$

4.2. On the symmetry type graphs of 4-orbit maniplexes

It does not take long to realise that counting the number of possible symmetry type graphs with $k \geq 4$ vertices, and perhaps classifying them in a similar fashion as was done for 2 and 3 vertices, becomes considerably more difficult. In this section, we shall analyse symmetry type graphs with 4 vertices and determine how far a 4-orbit maniplex can be from being fully-transitive. We start with a simple lemma.

Lemma 3. *Let $\mathcal{M}$ be a 4-orbit $(n-1)$-maniplex and let $i \in \{0, \ldots, n-1\}$. Then $\mathcal{M}$ has one, two or three orbits of i-faces.*

Proof. By Proposition 1, the number of orbits on i-faces is equal to the number of connected components of $T^i(\mathcal{M})$. Since $T(\mathcal{M})$ is connected, and it has at most two edges of colour i, it follows that $T^i(\mathcal{M})$ has at most three connected components.

$\square$

If an $(n-1)$-maniplex $\mathcal{M}$ is not fully-transitive, there exists at least one $i \in \{0, \ldots, n-1\}$ such that $T^i(\mathcal{M})$ is disconnected. We shall divide the analysis of the types in three parts: when $T^i(\mathcal{M})$ has three connected components (two of them of one vertex and one with two vertices), when $T^i(\mathcal{M})$ has a connected component with one vertex and another connected component with three vertices, and finally when $T^i(\mathcal{M})$ has two connected components with two vertices each. Let v_1, v_2, v_3, v_4 be the vertices of $T(\mathcal{M})$.

Suppose that $T^i(\mathcal{M})$ has three connected components with v_2 and v_3 in the same component. Without loss of generality we may assume that $T(\mathcal{M})$ has edges $(v_1, v_2)_i$ and $(v_3, v_4)_i$. Let $k \in \{0, 1, \ldots, n-1\} \setminus \{i\}$ be the colour of an edge between v_2 and v_3. Since there is no edge of $T(\mathcal{M})$ between v_1 and v_4, Lemma 2 implies that there are at most two such possible k, namely $k = i-1$ and $k = i+1$. If $i \notin \{0, n-1\}$, $T(\mathcal{M})$ can have either both edges or exactly one of them, while if $i \in \{0, n-1\}$ there is one possible edge (see Figure 6).

Let us now assume that $T^i(\mathcal{M})$ has two connected components, one consisting of the vertex v_1 and the other one containing vertices v_2, v_3 and v_4. This means that the i-edge incident to v_1 is the unique edge that connects this vertex with the rest of the graph and, without loss of generality, $T(\mathcal{M})$ has the edge $(v_1, v_2)_i$. As with the previous case, Lemma 2 implies that an edge between v_2 and v_3 has colour either $i-1$ or $i+1$.

First observe that having either $(v_2, v_3)_{i-1}$ or $(v_2, v_3)_{i+1}$ in $T(\mathcal{M})$ immediately implies (by Lemma 2) that there is no edge between v_2 and v_4. Now, if both edges $(v_2, v_3)_{i-1}$ and $(v_2, v_3)_{i+1}$ are in $T(\mathcal{M})$, then an edge between v_3 and v_4 would have to have colour i, contradicting the fact that $T^i(\mathcal{M})$ has two connected components. Hence, there is exactly one edge between v_2 and v_3. It is now straightforward to see that $T(\mathcal{M})$ should be one of the graphs

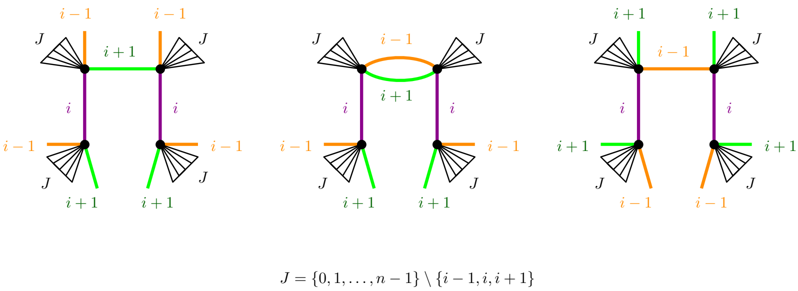


FIGURE 6. Symmetry type graphs of an $(n-1)$-maniplex $\mathcal{M}$ with four orbits on its flags, and three orbits on its i-faces.

in Figure 7, implying that there are four possible symmetry type graphs with these conditions for each $i \neq 0, 1, n-2, n-1$, but only two symmetry type graph of this kind when $i = 0, 1, n-2$, or $n-1$.

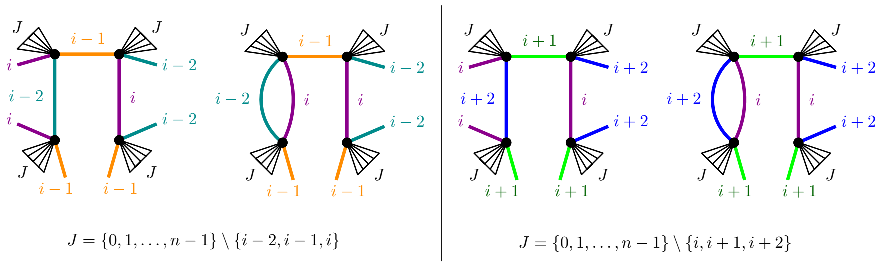


FIGURE 7. Symmetry type graphs of $(n-1)$-maniplexes with four orbits on its flags, and two orbits on its i-faces such that one contains three flag orbits and the other contains a single flag orbit.

It is straightforward to see from Figure 7 that the next lemma follows.

Lemma 4. *Let $\mathcal{M}$ be a 4-orbit $(n-1)$-maniplex with two orbits of i-faces such that $T^i(\mathcal{M})$ has a connected component consisting of one vertex, and another one consisting of three vertices. Then either $T^{i-1}(\mathcal{M})$ or $T^{i+1}(\mathcal{M})$ has two connected components, each with two vertices.*

Finally, we turn our attention to the case where $T^i(\mathcal{M})$ has two connected components, with two vertices each. Suppose that v_1 and v_2 belong to one component, while v_3 and v_4 belong to the other. As the two components must be connected by edges of colour i, we may assume that $(v_1, v_3)_i$ is an edge of $T(\mathcal{M})$. If the vertices v_2 and v_4 have semi-edges of colour i, Lemma 2 implies that $T(\mathcal{M})$ is one of the graphs shown in Figure 8. Note that if $i \in \{0, n-1\}$ there is one possible symmetry type graph for this particular case.

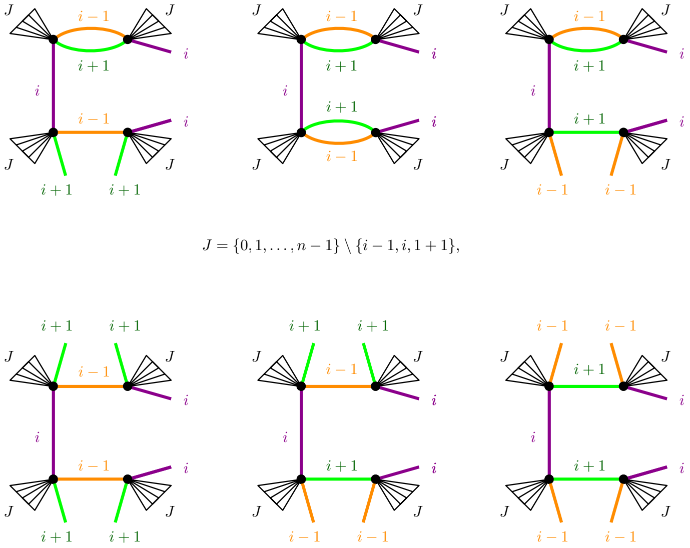
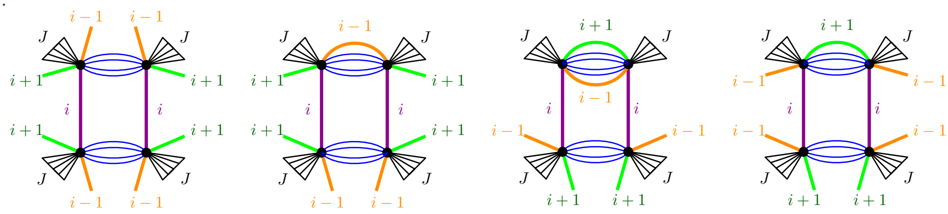


FIGURE 8. Six of the symmetry type graphs of $(n-1)$-maniplexs with four orbits on its flags, and two orbits on its i-faces such that each contains two flag orbits.

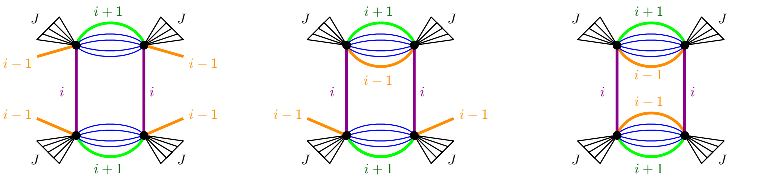
On the other hand, if $(v_1, v_3)_i$ and $(v_2, v_4)_i$ are both edges of $T(\mathcal{M})$, given $j \in \{0, 1, \ldots, n-1\} \setminus \{i-1, i, i+1\}$, we use again Lemma 2 to see that $(v_1, v_2)_j$ is an edge of $T(\mathcal{M})$ if and only if $(v_3, v_4)_j$ is also an edge of $T(\mathcal{M})$. By contrast, $T(\mathcal{M})$ can have either four semi-edges, an edge and two semi-edges, or two edges of colour $i \pm 1$ (each joining the vertices of each connected component of $T^i(\mathcal{M})$). Hence, if $i \neq 0, n-1$, for each $J \subset \{0, 1, \ldots, n-1\} \setminus \{i-1, i, i+1\}$ there are ten possible symmetry type graphs with four semi-edges of each of the colours in J and edges of colours not in J, as shown in Figures 9 and 10, while for $J = \{0, 1, \ldots, n-1\} \setminus \{i-1, i, i+1\}$ there are six such graphs (shown in Figure 10). On the other hand if $i \in \{0, n-1\}$, for each $J \subset \{0, 1, \ldots, n-1\} \setminus \{i-1, i, i+1\}$ there are two graphs as in Figure 9 and one as in Figure 10, while for $J = \{0, 1, \ldots, n-1\} \setminus \{i-1, i, i+1\}$, there is only one of the graphs in Figure 10.

Theorem 2 summarizes our analysis of the transitivity of 4-orbit maniplexes. In particular, case one holds if and only if $T(\mathcal{M})$ does not belong to any of the Figures 6 to 10. Moreover, case two holds if and only if $T(\mathcal{M})$ is either the graph in the middle of Figure 6, the graph in the middle in the first row of Figure 8, or any of the possible symmetry type graphs in Figures 9 and 10, whenever $J \subset \{0, 1, \ldots, n-1\} \setminus \{i-1, i, i+1\}$. Similarly, case four holds if and only if $T(\mathcal{M})$ is the graph in the middle of the second row of Figure 8. Finally, case three holds for all other possible choices of $T(\mathcal{M})$ (depicted in the first and third graphs of Figure 6, all graphs in Figure 7 and four of the six graphs in Figure 8).



$\emptyset \neq J \subset \{0, 1, \ldots, n-1\} \setminus \{i-1, i, 1+1\}$,

FIGURE 9. Four families of possible symmetry type graphs of $(n-1)$-maniplexes with four orbits on its flags, and two orbits on its i-faces such that each contains two flag orbits.



$\emptyset \neq J \subset \{0, 1, \ldots, n-1\} \setminus \{i-1, i, 1+1\}$,

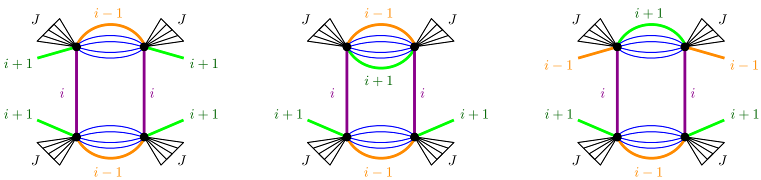


FIGURE 10. The remaining six families of possible symmetry type graphs of $(n-1)$-maniplexes with four orbits on its flags, and two orbits on its i-faces such that each contains two flag orbits.

Theorem 2. *Let $\mathcal{M}$ be a 4-orbit maniplex. Then, one of the following holds.*

1. *$\mathcal{M}$ is fully-transitive.*
2. *There exists $i \in \{0, \ldots, n-1\}$ such that $\mathcal{M}$ is j-face-transitive for all $j \neq i$.*
3. *There exist $i, k \in \{0, \ldots, n-1\}$, $i \neq k$, such that $\mathcal{M}$ is j-face-transitive for all $j \neq i, k$.*
4. *There exists $i \in \{0, \ldots, n-1\}$ such that $\mathcal{M}$ is j-face-transitive for all $j \neq i, i \pm 1$.*

4.3. On fully-transitive n-maniplexes for small n.

Every 1-maniplex is reflexible and hence fully-transitive. Fully-transitive 2-maniplexes correspond to fully-transitive maps. It is well-known (and easy to see from the symmetry type graph) that if a map is edge-transitive, then it should have one, two or four orbits of flags. Moreover, a fully-transitive map should be regular, a two-orbit map in class 2, 2_0, 2_1 or 2_2, or a four-orbit map in class 4_{Gp} or 4_{Hp} (see, for example, [4]).

When considering fully-transitive n-maniplexes, $n \geq 3$, the analysis becomes considerably more complicated. We deduce from [10] that there are at most $2^{n+1} - n - 2$ classes of fully-transitive 2-orbit n-maniplexes. By Theorem 1, there are no 3-orbit fully-transitive n-maniplexes. Extending the twenty-two possible symmetry type graphs of 4-orbit 2-maniplexes (see [4]) by adding (semi-) edges of colour 3 in such way that the $(0, 3)$ and $(1, 3)$ 2-factors are as in Figure 3, we note that there are twenty possible symmetry type graphs of 4-orbit 3-maniplexes that are fully transitive. We show these graphs in Figure 11.

Theorem 3. *Let $\mathcal{M}$ be a fully-transitive 3-maniplex and let $T(\mathcal{M})$ be its symmetry type graph. Then either $\mathcal{M}$ is reflexible or $T(\mathcal{M})$ has an even number of vertices.*

Proof. On the contrary suppose that $T(\mathcal{M})$ has an odd number of vertices, different than 1. Whenever $|i - j| > 1$, the connected components of the (i, j) 2-factor of a symmetry type graph are as in Figure 3. Hence, there is a connected component of the $(0, 2)$ 2-factor of $T(\mathcal{M})$ with exactly one vertex v (and, hence, semi-edges of colours 0 and 2). The connectivity of $T(\mathcal{M})$ implies that there is a vertex v_1 adjacent to v in $T(\mathcal{M})$.

If v_1 is the only neighbour of v, then $T(\mathcal{M})$ has the edges $(v, v_1)_1$ and $(v, v_1)_3$ as otherwise $\mathcal{M}$ is not fully-transitive. Since the connected components of the $(0, 3)$ 2-factor of $T(\mathcal{M})$ are as in Figure 3, v_1 has a 0 coloured semi-edge. Because $T(\mathcal{M})$ has more than two vertices, the edge of v_1 of colour 2 joins v_1 to another vertex, say u. But removing the edge $(v_1, u)_2$ disconnects the graph, contradicting the fact that $\mathcal{M}$ is 2-face-transitive.

On the other hand, if v has more than one neighbour it has exactly two, say v_1 and u, and $T(\mathcal{M})$ has the two edges $(v, v_1)_1$ and $(v, u)_3$. This implies that the connected component of the $(1, 3)$ 2-factor containing v has four vertices: v, v_1, u and v_2. (Therefore $(v_1, v_2)_3$ and $(u, v_2)_1$ are edges of $T(\mathcal{M})$.) Using the $(0, 3)$ 2-factor one sees that u has a semi-edge of colour 0.

Now, if $(v_1, v_2)_0$ is an edge of $T(\mathcal{M})$ or there are semi-edges coloured 0 at v_1 and v_2, then the vertices v, v_1, v_2 and u are joined to the rest of $T(\mathcal{M})$ only by the edges of colour 2, implying that removing them disconnects $T(\mathcal{M})$ (there exists at least another vertex in $T(\mathcal{M})$ since it has an odd number of vertices), which is again a contradiction. On the other hand, if v_1 (or v_2) has an edge of colour 0 to a vertex v_3 then, by Lemma 2, v_2 (or v_1) has a 0-edge to a vertex v_4. Again, if $(v_3, v_4)_1$ is an edge of $T(\mathcal{M})$ or there are semi-edges coloured 1 at v_3 and v_4, since the number of vertices of the graph is odd, removing the edges of colour 2 will leave only the vertices $u, v, v_1, \ldots, v_4$ in one component, which is a contradiction. Proceeding now by induction on the number of vertices one can conclude that $T(\mathcal{M})$ cannot have an odd number of vertices □

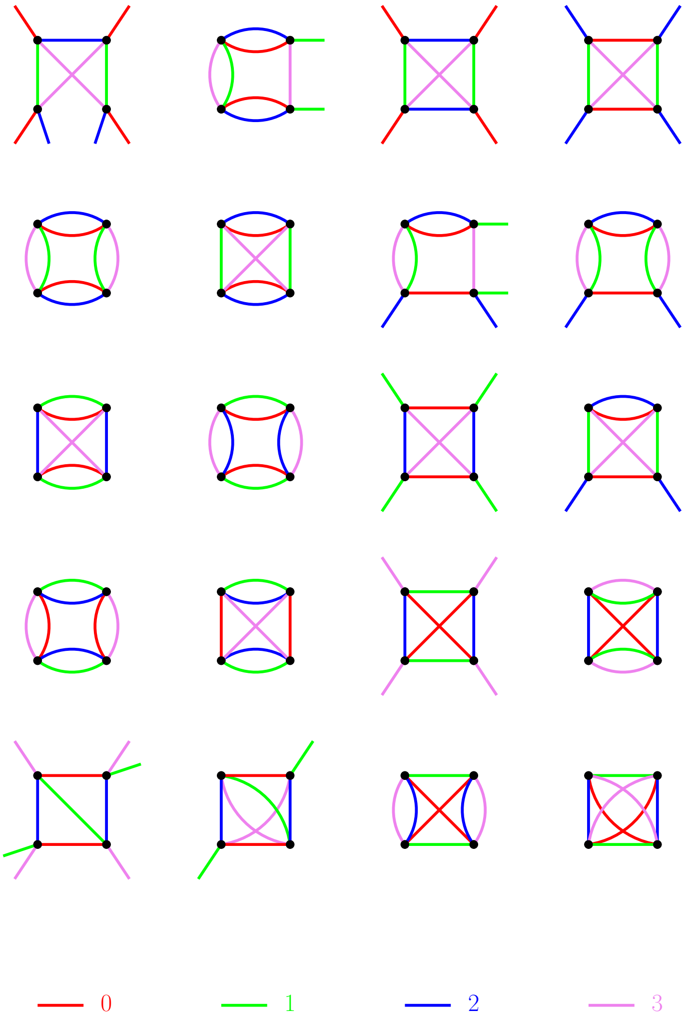


FIGURE 11. Symmetry type graphs of 4-orbit fully-transitive 3-maniplexes

5. Generators of the automorphism group of a k-orbit maniplex

It is well-known among polytopists that the automorphism group of a regular n-polytope can be generated by n involutions. In fact, given a base flag $\Phi \in \mathcal{F}(\mathcal{P})$, the distinguished generators of $\mathrm{Aut}(\mathcal{M})$ with respect to Φ are involutions $\rho_0, \rho_1, \ldots, \rho_{n-1}$ such that $\Phi\rho_i = \Phi^i$.

Generators for the automorphism group of a two-orbit n-polytope can also be given in terms of a base flag (see [10]). In this section we give a set of distinguished generators (with respect to some base flag) for the automorphism group of a k-orbit $(n-1)$-maniplex in terms of the symmetry type graph $T(\mathcal{M})$.

Given two walks w_1 and w_2 along the edges and semi-edges of $T(\mathcal{M})$ such that the final vertex of w_1 is the starting vertex of w_2, we define the sequence w_1w_2 as the walk that traces all the edges of w_1 and then all the edges of w_2 in the same order; the inverse of w_1, denoted by w_1^{-1}, is the walk which has the final vertex of w_1 as its starting vertex, and traces all the edges of w_1 in reversed order. Since each of the elements of $\mathrm{Mon}(\mathcal{M})$ associated to the edges of $T(\mathcal{M})$ is its own inverse, we shall forbid walks that trace the same edge two times consecutively (or just remove the edge from such walk, shortening its length by two). Given a set $\mathcal{W}$ of walks in $T(\mathcal{M})$, we say that a subset $\mathcal{W}' \subseteq \mathcal{W}$ is *a generating set of* $\mathcal{W}$ if each $w \in \mathcal{W}$ can be expressed as a sequence of elements of $\mathcal{W}'$ and their inverses. Now, let $\mathcal{W}$ be the set of closed walks along the edges and semi-edges of $T(\mathcal{M})$ starting at a distinguished vertex v_0. Recall that the walks along the edges and semi-edges of $T(\mathcal{M})$ correspond to permutations of the flags of $\mathcal{M}$; moreover, each closed walk of $\mathcal{W}$ corresponds to an automorphism of $\mathcal{M}$. Thus, by finding a generating set of $\mathcal{W}$, we will find a set of automorphisms of $\mathcal{M}$ that generates $\mathrm{Aut}(\mathcal{M})$. (However, the converse is not true, as an automorphism of $\mathcal{M}$ may be described in more than one way as a closed walk of $T(\mathcal{M})$.)

Given $T(\mathcal{M})$, the standard way to generate $\mathcal{W}$ is to consider a spanning tree $\mathcal{T}$ of $T(\mathcal{M})$. For each edge $e = (u, v)$ of $T(\mathcal{M})$ not in $\mathcal{T}$, we take as a generator the unique closed walk that consists of e, followed by the path from v to u in $\mathcal{T}$. Recall that if $\mathcal{N}$ is the normaliser of the stabiliser of a base flag Φ, then we may view $T(\mathcal{M})$ as the coset graph of $\mathcal{N}$ with respect to $\mathrm{Mon}(\mathcal{M})$. Viewing $T(\mathcal{M})$ this way, the closed walks we describe correspond to the Schreier generators of $\mathcal{N}$ in $\mathrm{Mon}(\mathcal{M})$ with respect to the generating set $\{r_0, \dots r_n\}$ (see for example [9]). In fact, the Schreier transversal corresponds precisely to the paths in the spanning tree $\mathcal{T}$ that start at the vertex $\mathcal{N}$. Hence, the Schreier generators $tr(\bar{tr})^{-1}$ such that t is in the transversal and $r = r_i$, $i = 0, \dots n$ correspond to the closed walks with at most one edge not in $\mathcal{T}$.

To have a standard presentation of the automorphism group of a maniplex, given its symmetry type graph, the above generating set is not convenient. For this reason, we give an alternative way of finding a generating set of closed walks, starting and finishing at a given vertex of the symmetry type graph. This generating set in turns will give rise to Theorem 4.

Our construction for a generating set of $\mathcal{W}$ goes as follows:

Let $\mathcal{M}$ be a k-orbit maniplex of rank $n-1$ and let $\mathcal{T}$ be a spanning tree of $T(\mathcal{M})$. The sets of vertices and edges (and semi-edges) of $T(\mathcal{M})$ will be denoted by V and E, respectively. The set of edges of $\mathcal{T}$ will be denoted by E_T. Let $\mathcal{C} = (v_0, e_0, v_1, e_1, ..., e_{k-1}, v_k)$ be a walk along the edges of $\mathcal{T}$ that visits all the vertices in E. Note that if $\mathcal{T}$ is not a single path, then $\mathcal{C}$ will visit some vertices in V and trace some edges in E more than once. That is, v_i may be equal to v_j for some $i, j \in \{0, ..., k-1\}$. Likewise, e_i may be equal to e_j for some $i, j \in \{0, ..., k-1\}$. We shall now construct $G(\mathcal{W}) \subseteq \mathcal{W}$, a generating set of $\mathcal{W}$.

For each edge $e \in E \setminus E_T$ between vertices v_i and v_j, $v_i \neq v_j$, we shall define the walk

$$w_{i,j,e} = ((v_0, v_1), (v_1, v_2), ..., (v_{i-1}, v_i), e, (v_j, v_{j-1}), (v_{j-1}, v_{j-2}), ..., (v_1, v_0)).$$

That is, we walk from v_0 to v_i in $E_{\mathcal{T}}$ tracing the same edges as $\mathcal{C}$, we take the edge e, and then we walk back from v_j to v_0 in $E_{\mathcal{T}}$ tracing the same edges as $\mathcal{C}$, but in reverse order. Let $\mathcal{W}_1 \subseteq \mathcal{W}$ be the set of all such walks.

For each semi-edge $s \in E \setminus E_{\mathcal{T}}$ we shall define the walk $w_{i,i,s} = ((v_0, v_1), (v_1, v_2), ..., (v_{i-1}, v_i), s, (v_i, v_{i-1}), (v_{i-1}, v_{i-2}), ..., (v_1, v_0))$. That is, we walk from v_0 to v_i in $E_{\mathcal{T}}$ tracing the same edges as $\mathcal{C}$, we take the semi-edge s, and then we walk back from v_i to v_0 in $E_{\mathcal{T}}$. Let $\mathcal{W}_2 \subseteq \mathcal{W}$ be the set of all such walks.

We define $G(\mathcal{W}) = \mathcal{W}_1 \cup \mathcal{W}_2$.

Lemma 5. *With the notation from above, $G(\mathcal{W})$ is a generating set for $\mathcal{W}$.*

Proof. We shall prove that any $w \in \mathcal{W}$ can be expressed as a sequence of elements of $G(\mathcal{W})$ and their inverses. Let $w \in \mathcal{W}$ be a closed walk along the edges and semi-edges of $T(\mathcal{M})$ starting at v_0. From now on, semi-edges will be referred to simply as "edges".

We shall proceed by induction over the number n of edges in $E \setminus E_{\mathcal{T}}$ visited by w. If w visits only one edge in $E \setminus E_{\mathcal{T}}$, then $w \in G(\mathcal{W})$ or $w^{-1} \in G(\mathcal{W})$. Let us suppose that, if a closed walk along the edges of $T(\mathcal{M})$ visits m different edges in $E \setminus E_{\mathcal{T}}$, with $m < n$, then it can be expressed as a sequence of elements of $G(\mathcal{W})$ and their inverses.

Let $w \in \mathcal{W}$ be a closed walk that visits exactly n edges in $E \setminus E_T$. Let $e \in E \setminus E_T$ be the last edge of $E \setminus E_T$ visited by w, and let v_a and v_b be the vertices incident to e. Without loss of generality we may assume that the vertex v_b was visited after v_a. Let f be the edge that w visits just before e, and let v_c and v_a be the vertices incident to f. That is, $f = (v_c, v_a)$ (note that f may or may not be in $E_{\mathcal{T}}$). Let $w_1 \in \mathcal{W}$ be the closed walk that traces the same edges (in the same order) as w until reaching f and then traces the edges (v_a, v_{a-1}), (v_{a-1}, v_{a-2}), ...,(v_1, v_0), and let $w_2 \in \mathcal{W}$ be the closed walk that traces the edges $(v_0, v_1), (v_1, v_2), ..., (v_{a-1}, v_a)$ and then traces e and continues the way w does to return to v_0. It is clear that w_1 visits exactly $n-1$ edges in $E \setminus E_{\mathcal{T}}$ and that w_2 visits only one. By inductive hypothesis both w_1 and w_2 can be expressed as a sequence of elements of $G(\mathcal{W})$, and therefore so does w since $w = w_1 w_2$. □

Let Φ be a base flag of $\mathcal{M}$ that projects to the initial vertex of a walk that contains all vertices of $T(\mathcal{M})$ of a symmetry type graph. Following the notation of [12], given $w \in \mathrm{Mon}(\mathcal{M})$ such that Φ^w is in the same orbit as Φ

(that is, $w \in \mathrm{Norm}(\mathrm{Stab}(\Phi))$), we denote by α_w the automorphism taking Φ to Φ^w. Moreover, if $w = r_{i_1} r_{i_2} \dots r_{i_k}$ for some $i_1, \dots i_k \in \{0, \dots, n-1\}$, then we may also denote α_w by $\alpha_{i_1, i_2, \dots i_k}$.

The following theorem gives distinguished generators (with respect to some base flag) of the automorphism group of a maniplex $\mathcal{M}$ in terms of a distinguished walk of $T(\mathcal{M})$, that travels through all the vertices of $T(\mathcal{M})$. Its proof is a consequence of the previous lemma.

Theorem 4. *Let $\mathcal{M}$ be a k-orbit n-maniplex and let $T(\mathcal{M})$ its symmetry type graph. Let $\mathcal{T}$ be a spanning tree for $T(\mathcal{M})$. Suppose that $v_1, e_1, v_2, e_2 \dots, e_{q-1}, v_q$ is a distinguished walk that visits every vertex of $T(\mathcal{M})$ tracing only edges of $\mathcal{T}$, with the edge e_i having colour a_i, for each $i = 1, \dots q-1$. Let $S_i \subset \{0, \dots, n-1\}$ be such that v_i has a semi-edge of colour s if and only if $s \in S_i$. Let $B_{i,j} \subset \{0, \dots, n-1\}$ be the set of colours of the edges between the vertices v_i and v_j (with $i < j$) that are not in the distinguished walk and let $\Phi \in \mathcal{F}(\mathcal{M})$ be a base flag of $\mathcal{M}$ such that Φ projects to v_1 in $T(\mathcal{M})$. Then, the automorphism group of $\mathcal{M}$ is generated by the union of the sets*

$$\{\alpha_{a_1, a_2, \dots, a_i, s, a_i, a_{i-1}, \dots, a_1} \mid i = 1, \dots, k-1, s \in S_i\},$$

and

$$\{\alpha_{a_1, a_2, \dots, a_i, b, a_j, a_{j-1}, \dots, a_1} \mid i, j \in \{1, \dots, k-1\}, i < j, b \in B_{i,j}\}.$$

We note that, in general, a set of generators of $\mathrm{Aut}(\mathcal{M})$ obtained from Theorem 4 can be reduced since there might be more than one element of $G(\mathcal{W})$ representing the same automorphism. For example, the closed walk w through an edge of colour 2, then a 0-semi-edge and finally a 2-edge corresponds to the element $r_2 r_0 r_2 = r_0$ of $\mathrm{Mon}(\mathcal{M})$. Hence, the group generator induced by the walk w is the same as that induced by the closed walk consisting only of the semi-edge of colour 0.

The following two corollaries give a set of generators for 2- and 3-orbit maniplexes, respectively, in a given class. The notation follows that of Theorem 4, where if the indices of some α do not fit into the parameters of the set, we understand that such automorphism is the identity.

Corollary 1. [11] *Let $\mathcal{M}$ be a 2-orbit $(n-1)$-maniplex in class 2_I, for some $I \subset \{0, \dots, n-1\}$ and let $j_0 \notin I$. Then*

$$\left\{\alpha_i, \alpha_{j_0, i, j_0}, \alpha_{k, j_0} \mid i \in I,\ k \notin I\right\}$$

is a generating set for $\mathrm{Aut}(\mathcal{M})$, *with respect to some base flag* Φ.

Corollary 2. *Let $\mathcal{M}$ be a 3-orbit $(n-1)$-maniplex.*

1. *If $\mathcal{M}$ is in class 3^i, for some $i \in \{1, \dots, n-2\}$, and Φ is a base flag of $\mathcal{M}$ that projects to the vertex on the right of the corresponding graph in Figure 5, then*

   $$\left\{\alpha_j, \alpha_{i,i-1,i+1,i}, \alpha_{i,i+1,j,i+1,i}, \alpha_{i,i+1,i,i+1,i} \mid j \in \{0, \dots, n-1\} \setminus \{i\}\right\}$$

   is a generating set for $\mathrm{Aut}(\mathcal{M})$, *with respect to the base flag* Φ.

2. *If $\mathcal{M}$ is in class $3^{i,i+1}$, for some $i \in \{0,\dots,n-2\}$, and Φ is a base flag of $\mathcal{M}$ that projects to the vertex on the right of the corresponding graph in Figure* 5*, then*

$$\{\alpha_j, \alpha_{i,j,i}, \alpha_{i,i+1,j,i+1,i}, \alpha_{i,i+1,i,i+1,i} \mid j \in \{0,\dots,n-1\} \setminus \{i\}\}$$

is a generating set for $\mathrm{Aut}(\mathcal{M})$*, with respect to the base flag* Φ*.*

6. Oriented and orientable maniplexes

A maniplex $\mathcal{M}$ is said to be *orientable* if its flag graph $\mathcal{G}_\mathcal{M}$ is a bipartite graph. Since a subgraph of a bipartite graph is also bipartite, all the sections of an orientable maniplex are orientable maniplexes themselves. An *orientation* of an orientable maniplex is a colouring of each part of $\mathcal{G}_\mathcal{M}$ with a different colour, say black and white. An *oriented maniplex* is an orientable maniplex with a given orientation. Note that any oriented maniplex $\mathcal{M}$ has an enantiomorphic maniplex (or mirror image) $\mathcal{M}^{en}$. One can think of the enantiomorphic form of an oriented maniplex simply as the orientable maniplex with the opposite orientation.

If the connection group $\mathrm{Mon}(\mathcal{M})$ of $\mathcal{M}$ is generated by $r_0, r_1, \dots, r_{n-1}$, for each $i \in \{0,\dots,n-2\}$ let us define the element $t_i := r_{n-1}r_i \in \mathrm{Mon}(\mathcal{M})$. Then, $t_i^2 = 1$, for $i = 0,\dots n-3$. The subgroup $\mathrm{Mon}^+(\mathcal{M})$ of $\mathrm{Mon}(\mathcal{M})$ generated by $t_0,\dots t_{n-2}$ is called *even connection group of* $\mathcal{M}$. Note that $\mathrm{Mon}^+(\mathcal{M})$ has index at most two in $\mathrm{Mon}(\mathcal{M})$. In fact $(\mathrm{Mon}^+(\mathcal{M}))^{r_{n-1}} = \mathrm{Mon}^+(\mathcal{M}^{en})$. It should be clear then that any oriented maniplex and its enantiomorphic form are in fact isomorphic as maniplexes.

An *oriented flag di-graph* ${\mathcal{G}_\mathcal{M}}^+$ of an oriented maniplex $\mathcal{M}$ is constructed in the following way. The vertex set of ${\mathcal{G}_\mathcal{M}}^+$ consists of one of the parts of the bipartition of $\mathcal{G}_\mathcal{M}$. That is, the black (or white) vertices of the flag graph of $\mathcal{M}$. The darts of ${\mathcal{G}_\mathcal{M}}^+$ will be the 2-arcs of $\mathcal{G}_\mathcal{M}$ of colours $n-1, i$, for each $i \in \{0,\dots,n-2\}$. We then identify two darts to obtain an edge if they have the same vertices, but go in opposite directions. Note that for $i = 0,\dots,n-3$ and each flag Φ of $\mathcal{M}$, the 2-arc starting at Φ and with edges coloured $n-1$ and i has the same end vertex than the 2-arc starting at Φ and with edges coloured i and $n-1$. Hence, all the darts corresponding to 2-arcs of colours $n-1$ and i, with $i = 0,\dots n-3$ will have both directions in ${\mathcal{G}_\mathcal{M}}^+$ giving us, at each vertex, $n-2$ different edges. On the other hand, the 2-arcs on edges of two colours $n-1, n-2$ will in general be directed darts of ${\mathcal{G}_\mathcal{M}}^+$. An example of an oriented flag di-graph is shown in Firgure 6. We note that the oriented flag di-graph of $\mathcal{M}^{en}$ can be obtained from ${\mathcal{G}_\mathcal{M}}^+$ by reversing the directions of the $n-2, n-1$ darts.

Note that the 2-arcs of colours r_{n-1}, r_i correspond to the generators t_i of $\mathrm{Mon}^+(\mathcal{M})$. In fact, as $\mathrm{Mon}^+(\mathcal{M})$ consists precisely of the even words of $\mathrm{Mon}(\mathcal{M})$, a maniplex is orientable if and only if the index of $\mathrm{Mon}^+(\mathcal{M})$ in $\mathrm{Mon}(\mathcal{M})$ is exactly two. We can then colour the edges and darts of ${\mathcal{G}_\mathcal{M}}^+$ with the elements t_i. The fact that $t_i^2 = 1$ for every $i = 0,\dots,n-3$ indeed implies

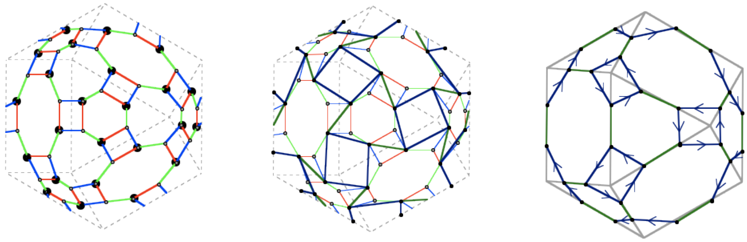


FIGURE 12. The oriented flag di-graph of an oriented cuboctahedron from its flag graph.

that the edges of ${\mathcal{G}_\mathcal{M}}^+$ are labelled by these first $n-2$ elements, while the darts are labelled by t_{n-2}.

We can see now that for each $i \in \{0, \dots, n-2\}$, the i-faces of $\mathcal{M}$ are in correspondence with the connected components of the subgraph of ${\mathcal{G}_\mathcal{M}}^+$ with edges of colours $\{0, \dots, n-2\} \setminus \{i\}$. To identify the facets of $\mathcal{M}$ as subgraphs of ${\mathcal{G}_\mathcal{M}}^+$, we first consider some oriented paths on the edges of ${\mathcal{G}_\mathcal{M}}^+$. We shall say that an oriented path on the edges of ${\mathcal{G}_\mathcal{M}}^+$ is *facet-admissible* if no two darts of colour t_{n-2} are consecutive on the path. Then, we will identify vertices of ${\mathcal{G}_\mathcal{M}}^+$ with the same facet of $\mathcal{M}$ if there exists a facet admissible oriented path from one of the vertices to the other.

For the remainder of this section, by a maniplex we shall mean an oriented maniplex, with one part of the flags coloured with black and the other one in white.

An *orientation preserving automorphism* of an (oriented) maniplex $\mathcal{M}$ is an automorphism of $\mathcal{M}$ that sends black flags to black flags and white flags to white flags. An *orientation reversing automorphism* is an automorphism that interchanges black and white flags. A *reflection* is an orientation reversing involutory automorphism. The group of orientation preserving automorphisms of $\mathcal{M}$ shall be denoted by $\mathrm{Aut}^+(\mathcal{M})$.

The orientation preserving automorphism $\mathrm{Aut}^+(\mathcal{M})$ of a maniplex $\mathcal{M}$ is a subgroup of index at most two in $\mathrm{Aut}(\mathcal{M})$. In fact, the index is exactly two if and only if $\mathrm{Aut}(\mathcal{M})$ contains an orientation reversing automorphism. Note that in this case, there exists an orientation reversing automorphism that sends $\mathcal{M}$ to its enantiomorphic form $\mathcal{M}^{en}$.

Pisanski [18] defines a maniplex to be *chiral-a-la-Conway* if $\mathrm{Aut}^+(\mathcal{M}) = \mathrm{Aut}(\mathcal{M})$. If a maniplex $\mathcal{M}$ is chiral-a-la-Conway, then its enantiomorphic maniplex $\mathcal{M}^{en}$ is isomorphic to $\mathcal{M}$, but there is no automorphism of the maniplex sending one to the other. It follows from the definition that $\mathcal{M}$ is chiral-a-la-Conway if and only if all automorphisms of $\mathcal{M}$ preserve the bipartition of $\mathcal{G}_\mathcal{M}$ and therefore we have the following proposition.

Proposition 6. *Let $\mathcal{M}$ be an oriented maniplex and let $T(\mathcal{M})$ its symmetry type graph. Then, $\mathcal{M}$ is chiral-a-la-Conway if and only if $T(\mathcal{M})$ has no odd cycles or semi-edges.*

Similarly as before, the orientation preserving automorphisms of a maniplex $\mathcal{M}$ correspond to colour preserving automorphism of the bipartite graph $\mathcal{G}_{\mathcal{M}}$ that preserves the two parts. But these correspond to colour preserving automorphisms of the di-graph ${\mathcal{G}_{\mathcal{M}}}^+$, implying that $\mathrm{Aut}^+(\mathcal{M}) \cong \mathrm{Aut}_p({\mathcal{G}_{\mathcal{M}}}^+)$. Note that the action of $\mathrm{Aut}^+(\mathcal{M})$ on the set $\mathcal{B}(\mathcal{M})$ of all the black flags of $\mathcal{M}$ is semiregular, and hence, the action on $\mathrm{Aut}_p({\mathcal{G}_{\mathcal{M}}}^+)$ is semiregular on the vertices of ${\mathcal{G}_{\mathcal{M}}}^+$.

An oriented maniplex $\mathcal{M}$ is said to be *rotary (or orientably regular)* if the action of $\mathrm{Aut}^+(\mathcal{M})$ is regular on on the vertices of ${\mathcal{G}_{\mathcal{M}}}^+$. Equivalently, $\mathcal{M}$ is rotary if the action of $\mathrm{Aut}_p({\mathcal{G}_{\mathcal{M}}}^+)$ is regular on its vertices. We say that $\mathcal{M}$ is *orientably k-orbit* if the action of $\mathrm{Aut}_p({\mathcal{G}_{\mathcal{M}}}^+)$ has exactly k orbits on the vertices. The following lemma is straightforward.

Lemma 6. *Let $\mathcal{M}$ be a chiral-a-la-Conway maniplex. Then $T(\mathcal{M})$ has no semi-edges, and if $\mathcal{M}$ is an orientably k-orbit maniplex, then $\mathcal{M}$ is a $2k$-orbit maniplex.*

6.1. Oriented symmetry type di-graphs of oriented maniplexes

We now consider the semiregular action of $\mathrm{Aut}^+(\mathcal{M})$ on the vertices of ${\mathcal{G}_{\mathcal{M}}}^+$, and let $\mathcal{B} = \mathcal{O}rb^+$ be the partition of the vertex set of ${\mathcal{G}_{\mathcal{M}}}^+$ into the orbits with respect to the action of $\mathrm{Aut}^+(\mathcal{M})$. (As before, since the action is semiregular, all orbits are of the same size.) The *oriented symmetry type di-graph* $T^+(\mathcal{M})$ of $\mathcal{M}$ is the quotient coloured di-graph of ${\mathcal{G}_{\mathcal{M}}}^+$ with respect to $\mathcal{O}rb^+$. Similarly as before, if $\mathcal{M}$ is rotary, then the oriented symmetry type di-graph of $\mathcal{M}$ consists of one vertex with one loop and $n-2$ semi-edges. Note that for oriented symmetry type di-graphs we shall not identify two darts with the same vertices, but different directions.

If we now turn our attention to oriented symmetry type di-graphs with two vertices, one can see that for each $I \subset \{0, \dots, n-2\}$, there is an oriented symmetry type di-graph with two vertices having semi-edges (or loops) of colours i at each vertex for every $i \in I$, and having edges (or both darts) of colour j, for each $j \notin I$. An oriented maniplex with such oriented symmetry type di-graph shall be said to be in class 2_I^+. Hence, there are $2^{n-2}-1$ possible classes of oriented 2-orbit $(n-1)$-maniplexes.

Note that if $\mathcal{M}$ is a k-orbit maniplex, then $T^+(\mathcal{M})$ has either k or $\frac{k}{2}$ vertices. The next result follows from Proposition 6 and Lemma 6.

Theorem 5. *Let $\mathcal{M}$ be an oriented maniplex. Then, $T(\mathcal{M})$ and $T^+(\mathcal{M})$ have the same number of vertices if and only if $T(\mathcal{M})$ has a semi-edge or an odd cycle.*

It is not difficult to see that if we are to consider for a moment an oriented symmetry type di-graph T^+, then the construction of Section 5 gives us a way to construct a generating set of the oriented closed walks based at the starting vertex of a oriented path containing all vertices (and Lemma 5 implies that the set actually generates.) Hence, one can find generators for the group of orientation preserving automorphisms of an oriented maniplex.

In particular we have the following theorem. Note that, in a similar fashion as before, we omit writing all generators obtained from Lemma 5, as many of them act in fact in the same fashion (for example, we do not include $\alpha_{n-2,n-1,i,n-1,n-1,n-2}$ as it is the same as automorphism as $\alpha_{n-2,n-1,i,n-2}$.)

Theorem 6. *Let $\mathcal{M}$ be an oriented 2-orbit $(n-1)$-maniplex in class 2_I^+, for some $I \subset \{0, \dots, n-2\}$. Then*

1. *If $n-2 \in I$, let $j_0 \notin I$, then*
$$\left\{\alpha_{i,n-1}, \alpha_{j_0,n-1,i,j_0}, \alpha_{k,n-1,j_0,n-1} \mid i \in I \; k \notin I\right\}$$
is a generating set for $Aut^+(\mathcal{M})$.
2. *If $n-2 \notin I$ but there exists $j_0 \notin I$, $j_0 \neq n-2$, then*
$$\left\{\alpha_{i,n-1}, \alpha_{j_0,n-1,i,j_0}, \alpha_{k,n-1,j_0,n-1}, \alpha_{n-1,n-2,j_0,n-1}, \alpha_{n-1,n-2,n-1,n-2} \mid i \in I \; k \notin I\right\}$$
is a generating set for $Aut^+(\mathcal{M})$.
3. *If $I = \{0, \dots, n-3\}$, then*
$$\left\{\alpha_{i,n-1}, \alpha_{n-2,n-1,i,n-1,n-2,n-1}, \alpha_{n-1,n-2,n-1,n-2} \mid i \in I\right\}$$
is a generating set for $Aut^+(\mathcal{M})$.

Given an oriented maniplex $\mathcal{M}$ and its symmetry type graph $T(\mathcal{M})$, we shall say that $T^+(\mathcal{M})$ is the associated oriented symmetry type di-graph of $T(\mathcal{M})$. Hence, given a symmetry type graph T one can find its associated oriented symmetry type di-graph T^+ by erasing all edges of T and replacing them by the $n-1, i$ paths of T. Note that this replacement of the edges may disconnect the new graph. If that is the case, we take T^+ to be one of the connected components.

6.2. Oriented symmetry type graphs with three vertices

In a similar way as one can classify maniplexes with small number of flag orbits (under the action of the automorphism group of the maniplex) in terms of their symmetry type graph, one can classify oriented maniplexes with small number of flags (under the action of the orientation preserving automorphism group of the maniplex) in terms of their oriented symmetry type di-graph.

We will now classify maniplexes with oriented symmetry type di-graphs with 3 vertices. Let $\mathcal{M}$ be a 6-orbit Chiral-a-la-Conway $(n-1)$-maniplex, with $n \geq 4$. Let $T(\mathcal{M})$ be its symmetry type graph and $T^+(\mathcal{M})$ be its oriented symmetry type di-graph. Recall that $T(\mathcal{M})$ is a graph with 6 vertices and no semi-edges or odd cycles, and that $T^+(\mathcal{M})$ is a di-graph with 3 vertices. Let $V = \{v_1, v_2, .., v_6\}$ be the vertex set of $T(\mathcal{M})$. We may label the vertices of $T(\mathcal{M})$ in such a way that the edges (v_1, v_2), (v_3, v_4), (v_5, v_6) are coloured with the colour $(n-1)$, and that the set $\{v_1, v_3, v_5\}$ constitutes one part of the bipartition. Let $\mathcal{W} = \{w_1, w_3, w_5\}$ be the vertex set of $T^+(\mathcal{M})$. Each $w_i \in \mathcal{W}$ corresponds to the vertex $v_i \in V$, $i \in \{1, 3, 5\}$. In what follows, in the same way as in Section 3, $(v_i, v_j)_k$ denotes the k-coloured edge joining the vertices v_i and v_j, $v_i, v_j \in V$, $k \in \{0, 1, ..., n-1\}$; and $(w_i, w_j)_k$ denotes the $(k, n-1)$-coloured edge joining the vertices w_i and w_j, $w_i, w_j \in \mathcal{W}$ and $k \in \{0, 1, ..., n-3)$.

Since there are no semi-edges in $T(\mathcal{M})$, for each colour $i \in \{0, ..., n-3\}$ there is one edge (and one semi-edge) of colour $(i, n-1)$ in $T^+(\mathcal{M})$ if and only if the 2-factor of $T(\mathcal{M})$ of colours i and $(n-1)$ consists of one 4-cycle and one 2-cycle of alternating colours. Likewise, there are three semi-edges of colour $(i, n-1)$ in $T^+(\mathcal{M})$ if and only if the 2-factor of $T(\mathcal{M})$ of colours i and $(n-1)$ consist of three 2-cycles. It is straightforward to see that there are two consecutive edges of colour $(i, n-1)$ and $(j, n-1)$, $i \neq j$, $i, j \leq n-3$, in $T^+(\mathcal{M})$ if and only if the 2-factor of colours i and j consists of a single 6-cycle. It follows that if there are two consecutive edges of colour $(i, n-1)$ and $(j, n-1)$ in $T^+(\mathcal{M})$, then $|i-j| < 2$.

Notice that the possible 2-factors of colour $(n-1)$ and $(n-2)$ in $T(\mathcal{M})$ are either a single 6-cycle of alternating colours, a 4-cycle along with a 2-cycle, or three separate 2-cycles. Hence, the darts in $T^+(\mathcal{M})$ are arranged in either a 3-cycle, a 2-cycle along with a loop, or three separate loops. We proceed case by case.

Consider the case when there are three loops in $T^+(\mathcal{M})$. Since oriented symmetry type di-graphs are connected, then without loss of generality $(w_1, w_3)_i$ and $(w_3, w_5)_{i+1}$ must be edges of $T^+(\mathcal{M})$. We may suppose that $(w_1, w_3)_i$ is the only edge joining w_1 and w_3. If there is a third edge in $T^+(\mathcal{M})$, then it is necessarily $(w_3, w_5)_{i-1}$. Note that, since the edges coloured by $(n-1)$ and $(n-2)$ do not lie on a 6-cycle in $T(\mathcal{M})$, there can be semi-edges of colour $0, 1, ..., n-3$ in $T^+(\mathcal{M})$. Thus, there is one oriented symmetry type di-graph for each pair of colours i and $i+1$, with $i \in \{0, ..., n-3\}$ and one for each triple $i-1$, i and $i+1$, $i \in \{1, ..., n-3\}$. Therefore, there are $2n-7$ oriented symmetry type di-graphs with 3 loops.

Consider the case when $T^+(\mathcal{M})$ has only one loop. We may suppose that the loop is in w_5 and the vertices w_1 and w_3 are joined by darts. This implies that $(v_1, v_4)_{(n-2)}$, $(v_2, v_3)_{(n-2)}$ and $(v_5, v_6)_{(n-2)}$ are edges of $T(\mathcal{M})$. As $T^+(\mathcal{M})$ is connected, there must be an edge joining w_3 and w_5 of colour $(i, n-1)$. Necessarily $i = n-3$, since the edges $(v_1, v_2)_i$, $(v_2, v_3)_{(n-2)}$, $(v_3, v_6)_i$, $(v_6, v_5)_{n-2}$, $(v_5, v_4)_i$, $(v_4, v_1)_{n-2}$ form a 6-cycle in $T(\mathcal{M})$. Hence, there are exactly two oriented symmetry type di-graph with a single loop: one with a single edge of colour $(n-3, n-1)$ between w_3 and w_5, and one with one edge of colour $(n-3, n-1)$ between w_3 and w_5, and one edge coloured $(n-4, n-1)$ between w_1 and w_3.

Consider the case when the darts in $T^+(\mathcal{M})$ are arranged in a 3-cycle. It is clear that the 2-factor of $T(\mathcal{M})$ of colours $(n-2)$ and $(n-1)$ is a single 6-cycle. Therefore, if $i \in \{0, ..., n-4\}$, the 2-factor $T(\mathcal{M})$ of colours i and $(n-1)$ cannot consist of three 2-cycles, as this implies the existence of a 6-cycle of alternating colours i and $(n-2)$ with $|i-(n-2)| \geq 2$. That is, $T^+(\mathcal{M})$ has one edge (and one semi-edge) of colour $(i, n-1)$ for each $i \in \{0, ..., n-4\}$ and either one edge and a semi-edge, or three semi-edges for colour $(n-3, n-1)$. Note that if $n \geq 7$, the set $\{0, ..., n-4\}$ has more than three elements and thus all edges of colour $(i, n-1)$ in $T^+(\mathcal{M})$, $i \in \{0, ..., n-4\}$, must be joining the same pair of vertices. Otherwise, there would be at least two consecutive edges of colours $(i, n-1)$ and $(j, n-1)$, with $|i-j| \geq 2$. Figure 6.2 below shows the only four possible oriented symmetry type di-graphs with a 3-cycle of darts and at least two consecutive edges. Two correpond to 4-maniplexes, one to 3-maniplexes and one to 5-maniplexes. (We note that as 2-maniplexes can be regarded as polygons, then they are always 1-orbit maniplexes.)

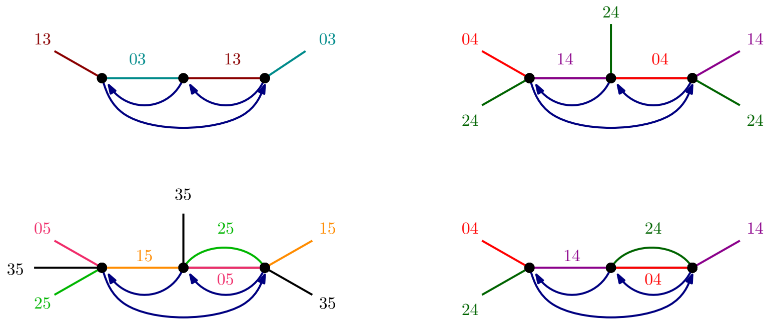


FIGURE 13. Oriented symmetry type di-graphs with 3 vertices and one directed 3-cylce, of 3-, 4- and 5-maniplexes

We may now suppose that $T^+(\mathcal{M})$ has no consecutive edges. It follows that here are exactly two oriented symmetry type di-graph with a 3-cycle of darts: one with an edge joining the same pair of vertices for each colour $i \in \{0, ..., n-3\}$, and one with three semi-edges of colour $(n-3, n-1)$ and an edge joining the same pair of vertices for each colour $i \in \{0, ..., n-4\}$.

Considering all the cases above, there are $(n-3)+(n-4)+2+2 = 2n-3$ oriented symmetry type graphs with three vertices for oriented maniplexes of rank $n \geq 6$; $2n-2 = 6$ for oriented maniplexes of rank 3; $2n-1 = 9$ for oriented maniplexes of rank 4; and $2n-2 = 10$ for oriented maniplexes of rank 5. Of course, given an orientable polytope, there are two possible oriented symmetry type graphs of the polytope, depending on the orientation. There two graphs will only depend on the orientation of the arcs.

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